Name: Solutions | Math 260
Start Time:
End Time:
Date:

Math 260

Quiz 5 (40 min)

1. (1 point) Complete the following definition:

Definition: The square matrix A is <u>invertible</u> if…

there exist a matrix B such that

$AB = I$ and $BA = I$.

2. (2 points) Prove the uniqueness of the inverse of a matrix. That is, prove the following statement:

If B and C are inverses of a square matrix A, then $B = C$.

proof: Suppose B and C are inverses of A.
Then $AB = I$, $BA = I$, $CA = I$ and $AC = I$.

Then $B = B \cdot I = B \cdot (AC) = (BA)C = IC = C$

So $B = C$.

3. (2 points) Use the matrix inversion algorithm to find A^{-1} if $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & 5 & -1 \end{bmatrix}$ (calculator OK!)

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 5 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & -\frac{1}{10} \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{9}{10} & \frac{7}{5} & -\frac{1}{10} \end{array}\right] = [I \mid A^{-1}]$$

$$\text{So } A^{-1} = \begin{bmatrix} \frac{1}{10} & \frac{2}{5} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ -\frac{9}{10} & \frac{7}{5} & -\frac{1}{10} \end{bmatrix}$$

4. (3 points) Write $A = \begin{bmatrix} 2 & 5 \\ -3 & -2 \end{bmatrix}$ as a product of elementary matrices.

$$\begin{bmatrix} 2 & 5 \\ -3 & -2 \end{bmatrix} \xrightarrow[\text{1}]{2R_1+R_2 \to R_2} \begin{bmatrix} 2 & 5 \\ 1 & 8 \end{bmatrix} \xrightarrow[\text{2}]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 8 \\ 2 & 5 \end{bmatrix} \xrightarrow[\text{3}]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 8 \\ 0 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 \\ 0 & -11 \end{bmatrix} \xrightarrow[\text{4}]{-\frac{1}{11}R_2 \to R_2} \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{5}]{-8R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E_1 \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_2 \qquad E_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_3 \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{11}R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{11} \end{bmatrix} = E_4 \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -11 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-8R_2+R_1 \to R_1} \begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix} = E_5 \qquad E_5^{-1} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$

$I = E_5 E_4 E_3 E_2 E_1 A \Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}$.

$$\boxed{\text{So } A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -11 \end{bmatrix}\begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}}$$

5. (2 points) Give an example of a square matrix that is not invertible and explain why it is not invertible

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, because if it had an inverse $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and could never equal the identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ no matter what you choose for a, b, c, d.

Extra Credit

1. (2 points) Prove: If A and B are $n \times n$ invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

proof: Let A and B be invertible. Then A^{-1} and B^{-1} exist.

Since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$,

AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

2. (2 points) Prove: If A is an $n \times n$ invertible matrix, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

proof: Suppose A is an invertible $n \times n$ matrix. Then A^{-1} exists.

$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$ and

$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

So A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

3. (4 points) True or False? (For each of the following, assume A and B are square matrices of the same size)

a) If A is an invertible matrix, then when reducing it to reduced row-echelon form, you get the identity matrix.

True

b) If the matrix A has a right inverse, then it is invertible.

True

c) If the system of equations $A\vec{x} = \vec{b}$ has a solution for any choice of $\vec{b}$ then A has an inverse.

True

d) If A and B are invertible then so is $A + B$

False. $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$ are both invertible bec. both have determinant $-1 \neq 0$. But $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible because it has determinant 0.